Analysis of Unsteady Transonic Channel Flow with Shock Waves

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Inviscid unsteady transonic flow in a two-dimensional channel is analyzed using asymptotic techniques. The analysis includes the case where a shock wave is present in a channel having arbitrary wall shape, with arbitrary small disturbances imposed at a given downstream location. Second-order solutions are not uniformly valid near the shock wave, since they do not satisfy the shock jump conditions. It is therefore necessary to obtain inner solutions which are matched asymptotically to those in the outer channel-flow region. Numerical results for an accelerating nozzle flow with a shock wave, where the nozzle back pressure has second-order sinusoidal oscillations, show the resulting shock-wave motion and unsteady flow downstream of the shock wave.

I. Introduction

An analysis of unsteady transonic channel flow with a shock wave has many important applications. For example, flows with shock waves may occur in the throat region of an inlet, in internal nozzle flows, or in the compressor of a jet engine. Unsteadiness of the flow in these applications can result from a variety of causes such as gusts, changes in engine power setting, bypass or bleed door actuation, and combustion or ignition pressure pulses.

Previously, solutions for unsteady transonic flow in two-dimensional channels have been presented in the form of similarity solutions. The similarity transformation given by Tomotiko and Tamada[1] for steady shockless flow, and employed by Sichel[2] for steady flows with thick shocks, has been extended to unsteady flows with shock waves by Adamson and Richey.[3] Although similarity transformations have the advantage of yielding relatively simple solutions with minimum computational effort and provide valuable insight into the nature of transonic channel flow, they do not provide a solution to the direct problem in which arbitrary wall shapes and initial conditions are specified. The solutions satisfy only special boundary conditions which may or may not correspond to a given physical problem. For example, in the cited[3] solutions, only special wall shapes can be considered, and the unsteadiness is associated with channel wall motion. Also, since the wall is instantaneously a streamline, the wall goes through a small change in slope at the wall-shock wave intersection. Hence, it is clear that for direct applications, different forms of solutions should be employed.

An approach to the direct problem for steady transonic channel flows has been discussed by Szaniawski,[4] who expanded the perturbation velocity potential in an assumed power series in the transverse coordinate, substituted this series into the general potential equation and boundary conditions for arbitrary wall shape, and found linear governing equations for the terms in the power series. It was later shown by Adamson, Messiter, and Richey[5] that this form of solution could be derived in a systematic fashion and that it was applicable to unsteady channel flows. Additional features appear when an infinitesimally thin shock wave exists in a steady channel flow described by this kind of solution. Messiter and Adamson[6] showed that, because the basic solutions did not satisfy shock jump conditions of second and higher order, it was necessary to consider an inner region containing the shock wave.

In this paper, solutions are presented for unsteady transonic channel flow with shock waves. The problem is formulated as a direct problem with unsteady flow; both the wall shape and initial conditions are arbitrary. Thus the previously mentioned solution[6] is extended here to cover unsteady flows. The characteristic time associated with the flow disturbances is chosen to fall in the so-called slowly varying time regime. The problem considered is that where the flow is steady in first order, with second-order oscillations in the channel back pressure causing oscillations in the position of a shock wave which exists in the channel, and in the flowfield downstream of the wave.

In the following, the flow is assumed to be two-dimensional, compressible, and transonic; the gas is assumed to follow the perfect-gas law and to have constant specific heats. The Reynolds number is taken to be large enough that viscous effects are negligible.

II. Governing Equations and Channel-Flow Solutions

The solution is written in terms of small, time-dependent perturbations from a uniform, steady, irrotational, sonic flow. Figure 1 shows the coordinate system and notation used. Velocity components U and V are made dimensionless with respect to the undisturbed sonic velocity, $\bar{a}^*$. (The overbar denotes a dimensional quantity.) Thermodynamic properties P and ρ are made dimensionless with respect to their

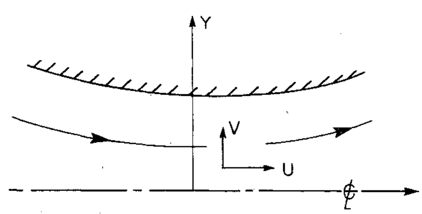

Fig. 1 Sketch of nozzle flow illustrating notation and coordinate system.

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values in the parent flow, $\bar{P}^*$ and $\bar{\rho}^*$, and the sound speed a is referred to $\bar{a}^*$. The dimensionless independent variables x, y, and T are made dimensionless with respect to the minimum channel half-width in the case of the space variables and with respect to $\bar{L}/\bar{a}^*$ in the case of time.

The region under consideration extends a distance from the throat which is of the order of the channel half-width, so that $x=0(1)$ and $y=0(1)$ here. If the characteristic time, $\bar{T}_{ch}$, associated with the imposed flow disturbances is not of the same order of magnitude as the characteristic flow time, $\bar{L}/\bar{a}^*$, the appropriate time coordinate is $t=\bar{T}/\bar{T}_{ch}$, which may be defined in terms of a nondimensional parameter τ:

$$T=\tau t \quad \tau=\bar{T}_{ch}/(\bar{L}/\bar{a}^*) \tag{1a,b}$$

The order of magnitude of τ depends on the physical problem under consideration. There are three main cases[7]: $\tau<<1$, $\tau=0(1)$, and $\tau>>1$. Physically, these cases correspond to characteristic times of the imposed flow disturbance which are much smaller than, on the order of, or much larger than the time required for a disturbance to cross the transonic region in the x direction. In this paper, the so-called slowly varying time regime is considered, where $\tau>>1$. Physical problems corresponding to this time regime are those associated with nonuniform air inlet distributions, changes in engine power setting and some combustion-forced oscillations in jet engines, and starting and stopping flows in nozzles.

Since the flow is described in terms of small perturbations from a uniform stream, a small parameter, $E<<1$, may be introduced, such that E is of the order of magnitude of the typical deviation of the nondimensional flow velocity from its sonic value. Thus $U-1=0(E)$ for $x=0(1)$. It will be seen later that E^2 is proportional to the ratio of the channel half-width to the wall radius of curvature at the nozzle throat. The order of magnitude of τ will be chosen such that $\tau=0(E^{-1})$, with τ defined by

$$\tau=(\mathrm{k}E)^{-1} \tag{2}$$

where k is a constant of order unity.

The undisturbed flow is irrotational and boundary layer effects are negligible. In addition, the flow is transonic, so that shock waves are weak, and the shock curvature is of high enough order that gradients in entropy introduced by the shock are $O(E^4)$. Hence a velocity potential function $\Phi(x,y,T)\equiv\bar{\Phi}/\bar{L}\bar{a}^*$ may be introduced, at least to the order considered here. The potential is written in an asymptotic expansion in E as follows:

$$\Phi(x,y,T)=x+E\phi_1(x,y,t)+E^2\phi_2(x,y,t)+\ldots \tag{3}$$

so that

$$U=\Phi_x=1+E\phi_{1x}+E^2\phi_{2x}+\ldots=1+Eu_1+E^2u_2+\ldots \tag{4a}$$

$$V=\Phi_y=E\phi_{1y}+E^2\phi_{2y}+\ldots=Ev_1+E^2v_2+\ldots \tag{4b}$$

where the subscripts x and y denote partial derivatives. The pressure is expanded as follows:

$$P=1+EP_1+E^2P_2+\ldots \tag{5}$$

with similar expressions being written for the density and temperature.

The governing equation for Φ is the so-called gas dynamic equation, written for unsteady flow[8]:

$$(a^2-\Phi_x^2)\Phi_{xx}+(a^2-\Phi_y^2)\Phi_{yy}-\Phi_{TT}-2\Phi_{xy}\Phi_x\Phi_y$$
$$-2\Phi_x\Phi_{xT}-2\Phi_y\Phi_{yT}=0 \tag{6}$$

The dimensionless speed of sound, $a^2=P/\rho$, is obtained from the Bernoulli equation

$$\Phi_T+a^2/(\gamma-1)+\tfrac{1}{2}(\Phi_x^2+\Phi_y^2)=(\gamma+1)/2(\gamma-1) \tag{7}$$

where γ is the (constant) ratio of specific heats and the term on the right-hand side of the equation is a constant rather than a function of time to and including terms of order E^2.

If Eqs. (1-3) and (7) are substituted into Eq. (6), and terms with like orders in E are equated, one finds governing equations for each of the ϕ_n. Thus,

$$\phi_{1yy}=0 \tag{8a}$$

$$\phi_{2yy}=2k\phi_{1xt}+(\gamma+1)\phi_{1x}\phi_{1xx} \tag{8b}$$

$$\phi_{3yy}=k^2\phi_{1tt}+2k\phi_{2xt}+\mathrm{k}(\gamma-1)\phi_{1xx}\phi_{1t}+2k\phi_{1x}\phi_{1xt}$$
$$+(\gamma+1)(\phi_{2x}\phi_{1x})_x+(\gamma-1)\phi_{1x}\phi_{2yy}+\left(\frac{\gamma+1}{2}\right)\phi_{1x}^2\phi_{1xx} \tag{8c}$$

For symmetric channels, where $V=0$ on the centerline, to all orders, it is seen from Eq. (8a) that $\phi_1=\phi_1(x,t)$ so that $\phi_{1y}\equiv 0$. This result has been used in deriving Eqs. (8b) and (8c). Then Eq. (8b) may be integrated, to give

$$\phi_2=[(\gamma+1)\phi_{1x}\phi_{1xx}+2\mathrm{k}\phi_{1xt}]\frac{y^2}{2}+h(x,t)+yg(x,t) \tag{9}$$

where $h(x,t)$ and $g(x,t)$ are functions of integration. For a symmetric channel, $g=0$. Because it is necessary to apply third-order boundary conditions to find an equation for $h(x,t)$, as will be seen, ϕ_{3y} must be calculated, and can be found by integrating Eq. (8c):

$$\phi_{3y}=[\mathrm{k}^2\phi_{1tt}+2\mathrm{k}\phi_{2xt}+\mathrm{k}(\gamma-1)\phi_{1xx}\phi_{1t}+$$
$$+(\gamma+1)(\phi_{2x}\phi_{1x})_x+2\mathrm{k}\gamma\phi_{1xt}\phi_{1x}$$
$$+(\gamma^2-1+\frac{\gamma+1}{2})\phi_{1x}^2\phi_{1xx}]y+h_3(x,t) \tag{10}$$

If the wall boundary is written in general as $y_w(x)$ and the wall tangency condition for inviscid flow is applied, it is easy to show that corresponding to the expansion for Φ, y_w may be written in the following general form, for stationary walls:

$$y_w=\pm(1+E^2f(x)) \tag{11}$$

where $f(0)=0$, since $x=0$ is defined to be at the location of the minimum channel area (i.e., $y_w=1$ at $x=0$). Furthermore, as a result of applying the boundary conditions, one can show that

$$(\phi_{2y})_{y=\pm 1}=\pm f' \tag{12a}$$

$$(\phi_{3y})_{y=\pm 1}=\pm[\phi_{1x}\phi_{2y}]_{y=\pm 1} \tag{12b}$$

where the upper and lower signs refer to the upper and lower walls, respectively, and $f'=\mathrm{d}f/\mathrm{d}x$. Equation (12a) is seen to be the usual steady-state boundary condition. It should be noted that Eqs. (11) and (12) can be generalized[9] to include the case where the channel walls have a prescribed motion, and to include higher-order shape functions, if desired.

If ϕ_{2y} is calculated from Eq. (9), evaluated at the wall, and substituted into Eq. (12a), one finds the governing equation for $\phi_{1x}=u_1(x,t)$:

$$\frac{2k}{(\gamma+1)}u_{1t}+u_1u_{1x}=f'/(\gamma+1) \tag{13}$$

Next, if the same procedure is followed with ϕ_{3y}, using Eqs. (9), (10), and (12b), the governing equation for $h(x,t)$ is obtained. This equation can be integrated once with respect to x; the resulting equation is

$$\frac{2k}{(\gamma+1)}h_t+u_1h_x=-\left\{\frac{1}{6}u_1f''+\frac{(2\gamma-3)}{6}u_1^3\right.$$
$$\left.+\frac{k(\gamma-1)}{(\gamma+1)}\phi_{1t}u_1\right\}+\frac{k}{4}\left(\frac{3-\gamma}{1+\gamma}\right)\int_x(u_1^2)_t\mathrm{d}x+A(t) \tag{14}$$

where the choice of $A(t)$, the function of integration, will be discussed later, in a particular problem.

Equations (13) and (14) give the rate of change of u_1 and h along characteristics with slope

$$(\mathrm{d}t/\mathrm{d}x) = 2k/(\gamma+1)u_1 \tag{15}$$

The variation of u_1 and h along these characteristics is determined by the channel shape as prescribed by the right-hand side of Eqs. (13) and (14). It is seen that only one family of characteristics exists. As pointed out[5] this is a result of the fact that disturbances are carried downstream, relative to the flow, at an absolute velocity $U+a \cong 2$, whereas disturbances moving upstream relative to the flow are carried at a velocity $U-a$, where $|U-a| << 1$. It is the latter disturbances which are observed here; disturbances moving downstream cross the transonic region in a time small compared with the characteristic time considered here.

It should be noted that if the x-derivative of Eq. (13) is substituted into Eq. (9), the following simple solutions for symmetric channels can be written:

$$u_2 = \phi_{2x} = f'' \frac{y^2}{2} + h_x \tag{16a}$$

$$v_2 = \phi_{2y} = f' y \tag{16b}$$

These results were used in deriving Eq. (14).

The solutions to second order are completed by solving for u_1 and h_x, using Eqs. (13) and (14) for a given wall shape and given initial conditions. To first order, the solution corresponds to a one-dimensional unsteady channel flow, since $u_1 = u_1(x,t)$ and $v_1 = 0$. The second-order solutions [Eqs. (16)] introduce two-dimensional effects, and also include unsteady effects for U through $h(x,t)$. However V is still independent of time to second order. Also, for stationary walls, the two-dimensional and unsteady effects in u_2 are separated. Higher-order solutions could be calculated, using the same procedures described above, but will not be pursued here.

The form of solution derived in this section holds throughout a channel flowfield except in a thin region enclosing a shock wave and possibly in a thin region enclosing a sonic line. The detailed behavior of these outer solutions in the vicinity of a sonic line has been discussed in Refs. 5 and 6 for steady and unsteady flow. Also, their behavior near a shock wave has been analyzed[6] for steady flow. There it was shown that these outer solutions were not uniformly valid in the flowfield downstream of the shock wave. Thus, although the first-order solutions for the velocity components did satisfy the shock-wave jump conditions, the second-order solutions did not. Therefore, downstream of the shock wave, it was necessary to consider an inner solution which satisfies the jump conditions and also matches with the outer solutions.

Here, for unsteady flow, it is found that the same general problem formulation holds when a shock wave is present in the channel, in a region of axial extent of the order of the minimum channel width. That is, the first-order outer solutions satisfy the jump conditions identically, but the second-order solutions do not, so an inner solution is required. Because the flow is unsteady, the solutions in the inner region can be quite different from or similar to the solutions found for steady flow, depending upon the problem considered. The essential point to be considered in this regard is the magnitude of the distance the shock moves as a result of impressed flow fluctuations. If this distance is large, of the order of the channel width, the governing equations for the inner region must be written for a moving coordinate system. If, on the other hand, this distance is small, such that for example the amplitude of the motion is small compared to the extent of the inner region, a stationary coordinate system suffices. Moreover, the equations contain no time derivatives, the time dependency arising from the boundary conditions, and the solutions are similar to the steady-flow solutions. Before carrying the analysis any further, then, it is necessary to choose a specific problem; here the latter of the two cases described above is chosen for analysis. It is desired to consider the problem where a shock wave exists in an accelerating symmetric channel flow which is initially steady. At time $t=0$, this flow has impressed upon it a fluctuating back pressure. That is, at some point in the channel, the area goes to a constant value and the channel then connects to a plenum which is undergoing pressure oscillations. These oscillations travel upstream and eventually reach the shock, causing it to change position in an oscillatory manner. The flow upstream of the shock remains steady. In general, the flow upstream of the shock may be unsteady as a result of upstream disturbances as described in Ref. 9. In this paper the method of solution is illustrated with examples in which disturbances arise only downstream of the shock.

The case considered is that where the distance moved by the shock is small compared to the channel width, so that the shock wave undergoes a small amplitude oscillation about a steady-state position. In this event, as will be shown, the impressed oscillations are of second order in E and the flow is steady to first order. Hence, Eq. (13) may be integrated easily to give $u_1 = u_1(x)$, and only u_2 varies with time, through h, as seen in Eq. (16a).

In the following section, the inner region in the vicinity of the shock wave is analyzed. Next, the initial value problem for $h(x,t)$ is considered, and in Sec. V numerical computations are shown for a specific wall shape and initial condition.

III. Flowfield Near the Shock Wave

It will be shown later that the shock-wave jump conditions are not satisfied by the solutions derived in Sec. II, in second order; as mentioned previously, this indicates that an inner region enclosing the shock wave must be considered. In this inner region, where flow modifications caused by the shock-wave curvature must be accounted for, the flow accelerations are large enough that the linear Eqs. (8) are not valid, and the first-order governing equation is the nonlinear small-disturbance equation. For the slowly varying time regime, then, it can be shown that the proper form of the stretched x variable in the inner region is the same as that in the steady-flow case, $x^* = (x-X_S)E^{-1/2}$, where $X_S(y,T)$ is the instantaneous shock wave location. However, because in the present case the shock wave oscillates about a steady-state position, say $x = x_0 =$ constant, with the amplitude of this motion being small compared to the extent of the inner region, it is sufficient to consider a stretched x variable

$$x^* = (x-x_0)E^{-1/2} \tag{17}$$

with y and t unchanged. Thus, to the scale of the outer region the shock wave can be located to sufficient accuracy at $x = x_0$ (i.e., $x^* = 0$). Then, if the outer velocity components, e.g., $U = 1 + Eu_1 + E^2 u_2 + \ldots$, are expanded in a Taylor expansion about $x = x_0$, and Δx is written in terms of x^* using Eq. (17), it is seen that half powers of E arise, so that the following inner expansions for U and V are suggested:

$$U = 1 + Eu_1^* + E^{3/2}u_{3/2}^* + E^2 u_2^* + E^{5/2}u^*_{5/2} + \ldots \tag{18a}$$

$$V = E^2 v_{3/2}^* + E^{5/2} v_2^* + \ldots \tag{18b}$$

With the velocity components written as in Eqs. (18), expressions for the shock-wave position $X_s(y,T)$ and the wave velocity, $\partial X_s/\partial T$ may be derived.[9] The wave velocity, normal to the curved wave front, is determined by setting the Eulerian derivative of $x - X_s(y,T)$ equal to zero. This normal wave velocity has x and y components as follows:

$$U_s = \frac{\partial X_s}{\partial T}\left[1 + \left(\frac{\partial X_s}{\partial y}\right)^2\right]^{-1} \qquad V_s = -U_s \frac{\partial X_s}{\partial y} \tag{19a,b}$$

The instantaneous value of $-\partial X_s/\partial y$ at a given position equals the jump in V across the wave divided by the jump in U, written either in terms of absolute or relative (to the wave) velocities. Anticipating the result that the jump in $v_{3/2}$ is zero at the wave, it is seen from Eqs. (18) that $\partial X_s/\partial y = 0$ $(E^{3/2})$. Moreover, because the shock-wave position can be calculated more accurately as each order of approximation is considered in the general problem, one can write $X_s(y,T)$ in terms of an asymptotic expansion

$$X_s(y,T) = x_0 + E^\alpha x_{\alpha+1}(t) + E^{3/2} x_{5/2}(y,t) + \ldots \quad (20)$$

where, again, x_0 is the steady-state position and $x_{\alpha+1}$ is a function of time alone as long as $\alpha < 3/2$. Under this tentative condition, it is seen from Eqs. (1), (2), and (19a) that

$$U_s = kE^{\alpha+1} dx_{\alpha+1}/dt + \ldots \quad (21)$$

Now, the acceleration of the shock from rest is induced by disturbances from the outer region transmitted through the inner region. Thus, it is clear that the expansion for the shock velocity must proceed with the same sequence of orders of E as that found in the expansion for U, Eq. (18a). For the problem considered, where the shock wave oscillates about a given steady-state position with small amplitude, it is seen that $\alpha > 0$, so that $\alpha + 1 > 1$ in Eq. (21). Hence, referring to Eq. (18a), it is clear that for this problem, the flow is steady to first order. Next, anticipating the result, to be shown later, that $u^*_{3/2} = 0$ at the shock wave so that $U_s = 0$ to this order, it follows that since $u^*_2 \neq 0$ at the shock, $\alpha = 1$, and so

$$X_s(y,T) = x_0 + Ex_2(t) + E^{3/2} x_{5/2}(y,t) + \ldots \quad (22)$$

and U_s becomes

$$U_s = kE^2 \frac{dx_2}{dt} + \ldots = kE^2 u_{s2} + \ldots \quad (23)$$

From Eqs. (19b) and (22), one can show that $V_s = 0(E^{7/2})$ and so is not of importance to the order of approximation considered here.

With the aid of Eqs. (17, 18, 22, and 23), the mathematical problem in the inner region can be completely formulated. First from Eqs. (17) and (18), it follows that the velocity potential for the inner region may be written as follows:

$$\Phi^* = E^{1/2}[x^* + E\phi^*_1 + E^{3/2}\phi^*_{3/2} + E^2\phi^*_2 + \ldots] \quad (24)$$

The region under consideration is thin $[\Delta x = 0(E^{1/2})]$ and encloses the shock wave. If this potential function is substituted into the governing Eqs. (16) and (7) with the stretched x variable defined in Eq. (17), the governing equations for ϕ^*_n are found to be:

$$-(\gamma+1)\phi^*_{1x^*}\phi^*_{1x^*x^*} + \phi^*_{1yy} = 0 \quad (25a)$$

$$-(\gamma+1)[\phi^*_{(3/2)x^*}\phi^*_{1x^*}]_{x^*} + \phi^*_{(3/2)yy} - 2k\phi^*_{1x^*t} = 0 \quad (25b)$$

$$-(\gamma+1)[\phi^*_{2x^*}\phi^*_{1x^*}]_{x^*} - ((\gamma+1)/2)[(\phi^*_{1x^*})^2\phi^*_{1x^*x^*} + 2\phi^*_{(3/2)x^*}\phi^*_{(3/2)x^*x^*}] + \phi^*_{2yy} - (\gamma-1)\phi^*_{1x^*}\phi^*_{1yy} - 2\phi^*_{1y}\phi^*_{1x^*y} - 2k\phi^*_{(3/2)x^*t} = 0 \quad (25c)$$

Within the inner region, the solutions to these equations must satisfy the instantaneous shock-wave jump conditions, and as $x^* \to \pm\infty$ they must match with the outer solutions. It can be shown[9] that for the slowly varying time regime, the instantaneous shock jump conditions apply if they are written in terms of velocities relative to the shock wave. In this case, because the shock is moving, the stagnation enthalpy conserved across the wave is that associated with a moving coordinate system attached to the wave. Thus, the modified form of the shock polar used to derive the shock jump conditions in the inner region is, where the tilde is used to denote velocities relative to the shock wave, (e.g, $\tilde{U}^* = U^* - U_s$)

$$[\tilde{V}^*_d\tilde{U}^*_u - \tilde{U}^*_d\tilde{V}^*_u]^2 = \{\tilde{U}^{*2}_u + \tilde{V}^{*2}_u - \tilde{U}^*_d\tilde{U}^*_u - \tilde{V}^*_d\tilde{V}^*_u\}^2 \cdot [\tilde{U}^*_d\tilde{U}^*_u + \tilde{V}^*_d\tilde{V}^*_u - 1 + 2\left(\frac{\gamma-1}{\gamma+1}\right)kE^2u_{s2}]\left[\left(\frac{2}{\gamma+1}\right) \cdot (\tilde{U}^{*2}_u + \tilde{V}^{*2}_u) - (\tilde{U}^*_d\tilde{U}^*_u + \tilde{V}^*_d\tilde{V}^*_u - 1) - 2\left(\frac{\gamma-1}{\gamma+1}\right)kE^2u_{s2}\right]^{-1} \quad (26)$$

where subscripts u and d refer to conditions immediately upstream and downstream of the shock wave, respectively. Now Eq. (26) holds at the shock wave, $x^* = x^*_s$. However, from Eqs. (17) and (22), one finds that

$$x^*_s = (X_s - x_0)E^{-1/2} = E^{1/2}x_2 + Ex_{5/2} + \ldots \quad (27)$$

Hence, one can write the velocity components which have been evaluated at x^*_s in terms of expansions about $x^* = 0$, and then from Eq. (26) find equivalent jump conditions to be applied at $x^* = 0$, for each order of approximation. These jump conditions, which must be satisfied at any time, t, are

$$u^*_{1u} + u^*_{1d} = 0 \quad (28a)$$

$$u^*_{(3/2)u} + u^*_{(3/2)d} + x_2((u^*_{1x^*})_d + (u^*_{1x^*})_u) = 0 \quad (28b)$$

$$[v^*_{(3/2)d} - v^*_{(3/2)u}]^2 = 2(\gamma+1)u^{*2}_{1u}\{u^*_{2u} + u^*_{2d} + x_2[(u^*_{(3/2)x^*})_u + (u^*_{(3/2)x^*})_d] + \frac{x_2^2}{2}[(u^*_{1x^*x^*})_u + (u^*_{1x^*x^*})_d] + x_{5/2}[(u^*_{1x^*})_u + (u^*_{1x^*})_d] - \frac{4k}{(\gamma+1)}u_{s2} - (u^*_{1u})^2\} \quad (28c)$$

where now the subscripts u and d denote conditions at $x^* = 0^-$ and $x^* = 0^+$ (upstream and downstream of the shock) respectively. Thus although the shock is moving, for the case considered, it is possible to reduce the problem to an equivalent steady-state problem with the shock at $x^* = 0$.

The matching conditions to be satisfied as $x^* \to \pm\infty$ are derived from the outer solutions. Because the flow is steady to first order in the inner region and this solution must match with the first-order solution in the outer region, the flow is steady to first order everywhere. Therefore, Eq. (13) may be integrated immediately to give,

$$u_1 = \phi_{1x} = \pm\left(\frac{2}{(\gamma+1)}f(x) + c_w\right)^{1/2} \quad (29)$$

where the upper and lower signs refer to supersonic and subsonic flows, respectively. If, as is the case here, the flow is sonic at the throat, where $f(x) = 0$, then $c_w = 0$. At the shock, $f(x_0)$ is denoted by f_0 and u_1 is written as

$$u_1(x_0) = \pm\left(\frac{2}{\gamma+1}f_0\right)^{1/2} = \pm c_u \quad (30)$$

Finally, one can expand the outer solutions for U and V[Eqs. (4a) and (4b), with Eq. (29) for u_1, $v_1 = 0$, and Eqs. (16a) and (16b) for u_2 and v_2] about $x = x_0$ and write these expressions in terms of x^*. The results are

$$U = 1 \pm Ec_u \pm E^{3/2}x^*f'_0/(\gamma+1)c_u + E^2\{\tfrac{1}{2}f''_0y^2 + h_x(x_0,t) \pm \frac{x^{*2}}{2(\gamma+1)} \cdot [f''_0/c_u - (f'_0)^2/(\gamma+1)c_u^3]\} + \ldots \quad (31a)$$

$$V=E^2f_0'y+E^{5/2}x^*f_0''y+\ldots \tag{31b}$$

Equations (31) are those to which the corresponding inner solutions must be matched term by term as $|x^*|\to\infty$.

The boundary conditions to be satisfied at the wall in the inner region are found easily by applying the wall tangency condition. They reduce to[9]

$$v_{3/2}^*(x^*,\pm 1)=\pm f_0' \tag{32a}$$

$$v_2^*(x^*,\pm 1)=\pm x^* f_0'' \tag{32b}$$

The problem in the inner region is solved in two steps. First the supersonic flow upstream of the shock wave ($x^*<0$) is considered. The governing equations are Eqs. (25) with initial conditions given by Eqs. (31) (upper signs) and boundary conditions given by Eqs. (32). These solutions are evaluated at $x^*=0$ and inserted into the shock jump conditions, Eqs. (28), thus giving the upstream boundary conditions ($x^*=0^+$) for the subsonic flow downstream of the shock. The boundary conditions for the subsonic boundary-value problem consist of those just mentioned at $x^*=0^+$, the boundary conditions at the walls, given by Eqs. (32) with $x^*>0$, and the matching conditions valid as $x^*\to\infty$, Eqs. (31) (lower signs). For the case considered, the problem formulation is, except for a few minor details, the same as that used in the steady-flow case[6] so only a brief outline of the methods used and the solutions obtained is given here. Details of the present calculations can be found in Ref. 9.

In the supersonic region upstream of the shock wave, it is found that the solutions which satisfy the governing equations and boundary conditions and which match term by term with the outer solutions (initial conditions) as $x^*\to-\infty$, are the outer solutions themselves. [Eq. (31) with upper signs.] That is, the inner solutions merely continue the outer solutions to the shock wave. Thus, from Eqs. (31a), $u_1^*=c_u$, $u_{3/2}^*=x^*f_0'/(\gamma+1)c_u$, etc. Downstream of the shock wave it is found that the solutions for ϕ_1^* and $\phi_{3/2}^*$ are also simply continuations of the outer solution in the inner region. Hence from Eq. (31), for $x^*>0$, $\phi_{1x^*}^*=u_1^*=-c_u$, $u_{3/2}^*=-x^*f_0'/(\gamma+1)c_u$, and $v_{3/2}^*=f_0'y$. If the solutions for ϕ_1^* and $\phi_{3/2}^*$, both upstream and downstream of the shock wave, are evaluated at $x^*=0$ and substituted into Eqs. (28a) and (28b), it is seen that the shock jump conditions are satisfied identically (note that since u_1^* = constant, $u_{1x^*}^*=0$) and that, as mentioned previously, $u_{s3/2}=0$ and the jump in $v_{3/2}^*$ across the shock is zero. However, it is also evident that the second-order jump condition, Eq. (28c), which with the previous solutions reduces to

$$u_{2d}^*=u_{1u}^{*\,2}+\frac{4k}{(\gamma+1)}u_{s2}-u_{2u}^* \tag{33}$$

is not satisfied by the second-order terms in Eq. (31a). Therefore, ϕ_2^* is not just a simple continuation of the outer solution.

The solution for ϕ_2^* is thus written as the sum of the continuation of the outer solution and an unknown potential ζ^*, i.e.

$$\phi_2^*=\frac{x^{*3}}{6(\gamma+1)}\left[(f_0')^2/(\gamma+1)c_u^3-f_0''/c_u\right]$$

$$+\frac{f_0''}{2}y^2x^*+h_{xd}x^*+\zeta^* \tag{34}$$

When this form for ϕ_2^* and the solutions for ϕ_1^* and $\phi_{3/2}^*$ are substituted into the governing equation (25c), the boundary conditions, Eq. (32b), the post-shock conditions at $x^*=0^+$ Eq. (33), and the matching conditions as $x^*\to\infty$ [term of order E^2 in Eq. (31a) with lower sign] one finds the following equation and boundary conditions for ζ^*.

$$(\gamma+1)c_u\zeta_{x^*x^*}^*+\zeta_{yy}^*=0 \tag{35a}$$

$$\zeta_{x^*}^*(x^*,1,t)=0 \quad \zeta_y^*(x^*,0,t)=0 \tag{35b,c}$$

$$\zeta_{x^*}^*(0,y,t)=c_u^2+\frac{4k}{(\gamma+1)}u_{s2}-f_0''y^2-(h_{xd}+h_{xu}) \tag{35d}$$

$$\lim_{x^*\to\infty}\zeta_{x^*}^*=0 \tag{35e}$$

That is, the problem is a boundary-value problem of the Neumann type. From the integral condition which must be met for a Neumann problem, it follows that the integral of $\zeta_{x^*}^*(0,y,t)$ from $y=0$ to $y=1$ must be zero. Thus

$$\frac{4k}{(\gamma+1)}u_{s2}=h_{xd}+h_{xu}+\frac{f_0''}{3}-c_u^2 \tag{36}$$

where it should be noted that u_{s2}, h_{xd}, and h_{xu} are all functions of time. Because the governing equations and boundary conditions involve only space derivatives here, they may be treated as constants in the computation for ζ^*.

The solution for ζ^* which satisfies the governing equation and boundary conditions, is

$$\zeta^*=\frac{4f_0''[(\gamma+1)c_u]^{1/2}}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}$$

$$\cdot\exp\{-n\pi x^*/[(\gamma+1)c_u]^{1/2}\}\cos(n\pi y) \tag{37}$$

from which the velocity components $\zeta_{x^*}^*$ and ζ_y^* can be calculated.

Downstream of the shock wave, a composite solution may be formed by adding the inner and outer solutions and subtracting the common terms (i.e., those used in matching solutions). Because the first differences between the inner and outer solutions occur in second-order terms, these composite solutions can be written in the following simple form:

$$U=1+E\phi_{1x}+E^2(\phi_{2x}+\zeta_{x^*}^*)+\ldots \tag{38a}$$

$$V=E^2(\phi_{2y}+\zeta_y^*)+\ldots \tag{38b}$$

where ϕ_{1x}, ϕ_{2x}, and ϕ_{2y} are given by Eqs. (29), (16a), and (16b), respectively. Upstream of the shock, the same relations hold, but with $\zeta_{x^*}^*=\zeta_y^*=0$. It is interesting to note that the shock-induced corrections to the outer-flow solutions, given by $\zeta_{x^*}^*$ and ζ_y^*, decay exponentially with increasing x^* [Cf. Eq. (37)]. Also, from Eq. (38), with ζ_y^* calculated from Eq. (37), one can show that there is a jump in v_2 everywhere along the shock wave except at $y=0$ and $y=1$, so that the wave has an inflection point and a shape which depends on f_0''.

In order to complete the solution for the velocity components u_2, it is necessary to compute h_x, as indicated in Eq. (16a). Once this has been accomplished, velocity components are known to second order everywhere in the flowfield. In addition, it should be noted that Eq. (36) allows a calculation of u_{s2} once the values of h_{xu} and h_{xd} are known. Thus, as a variation in h_{xu} and h_{xd} is experienced at the shock wave, a shock velocity u_{s2} is induced. The instantaneous shock position then is $X_s=x_{01}+Ex_2+\ldots$ where $x_2(t)$ is found by integrating u_{s2} with respect to time.

IV. Initial Value Problem for $h_x(x,t)$

Because the flow is steady to first order, i.e., because $u=u_1(x)$, the governing equation for $h(x,t)$ [Eq. (14)] reduces to:

$$(2k/(\gamma+1))h_t+u_1h_x=A(t)-u_1(x)F'(x) \tag{39a}$$

$$F'(x)=[f_{xx}+(2\gamma-3)u_1^2]/6 \tag{39b}$$

As noted previously, Eq. (39a) has characteristics with slope given by Eq. (15). One can define variables s, r such that $r=$

constant along characteristics and such that Eq. (39a) becomes

$$(\partial h/\partial s) = A - u_l F' \tag{40}$$

where

$$(\partial x/\partial s) = u_l \quad (\partial t/\partial s) = 2k/(\gamma+1) \tag{41a,b}$$

Now, if lines of constant s are chosen to be lines of constant t, moreover with $s=0$ at $t=0$, then Eq. (41b) is integrated as follows:

$$t = [2k/(\gamma+1)]s \tag{42}$$

In general, x is related to s and r through the integral of Eq. (41a) along a characteristic as follows:

$$x(r,s) = \int_0^s u_l ds + r \tag{43}$$

where $x(r,0)$ has been set equal to r. Also, by taking the derivative of Eq. (41a) with respect to r and then integrating over s, one finds that

$$\hat{g}(r,s) \equiv \frac{\partial x}{\partial r})_s = \exp\left[\int_0^s \frac{\partial u_l}{\partial x} ds\right] \tag{44}$$

Other relationships between the x,t and r,s coordinates are

$$\frac{\partial s}{\partial x} = 0, \; \frac{\partial s}{\partial t} = \frac{\gamma+1}{2k}, \; \frac{\partial r}{\partial x} = \frac{1}{\hat{g}}, \; \frac{\partial r}{\partial t} = -\left(\frac{\gamma+1}{2k}\right)\frac{u_l}{\hat{g}} \tag{45a-d}$$

Equation (40) may be integrated along a chosen r characteristic from $s=0$ (initial condition) to the s-value at which it is desired to evaluate $h(x,t) = \hat{h}(r,s)$.

$$\hat{h}(r,s) = -\int_0^s u_l F' ds + \int_0^s A ds + \hat{h}_0 \tag{46}$$

where $\hat{h}_0 = \hat{h}(r,0)$ is the initial condition which is modified as it is transmitted along a characteristic. The desired $(\partial h/\partial x)$ is found from Eq. (46) using Eqs. (45); thus

$$\frac{\partial h}{\partial x} = -\frac{1}{\hat{g}}\int_0^s \hat{g}(u_l F')' ds + \frac{1}{\hat{g}}\frac{d\hat{h}_0}{dr} \tag{47}$$

Equations (43, 44, 46, and 47) illustrate the general solution to the initial-value problem and would hold, for example, even if u_l were dependent upon time and thus $F=F(x,t)$. In the present case, since $u_l = u_l(x)$, it is possible to simplify the solution considerably. Thus, rather than using Eq. (43), one could write the integral of Eq. (41a) as

$$s = \int_r^x \frac{d\xi}{u_l(\xi)} \tag{48}$$

Then, differentiating Eq. (48) with respect to r, it is found that

$$\hat{g} = (\partial x/\partial r) = u_l(x)/u_l(r) \tag{49}$$

Finally, then, Eqs. (46) and (47) are found to be, since along a characteristic $ds = dx/u_l$,

$$h = -F(x) + \left(\frac{\gamma+1}{2k}\right)\int_0^t A dt + F(r) + \hat{h}_0(r) \tag{50a}$$

$$h_x = -F' + \frac{u_l(r)}{u_l(x)}\left[\frac{d\hat{h}_0}{dr} + F'(r)\right] \tag{50b}$$

Equation (50a) is seen to be the solution of Eq. (40) expressed as the sum of a particular solution written in terms of the original variable, x, t, and a solution to the homogeneous equation, $F(r) + \hat{h}_0(r)$.

Equations (42) and (48) can be used to determine the characteristics, r=constant, in the x,t plane for a given $u_l(x)$, which in turn depends upon the wall shape. From Eq. (15), it is seen that upstream of the shock wave ($u_l>0$) the characteristics have a positive slope and downstream of the shock ($u_l<0$), a negative slope. Typical characteristics for a channel with a parabolic wall shape are shown in Fig. 2.

It is seen from Eq. (47) or (50b) that h_x may be calculated along any characteristic and therefore at any position x and any time t for given initial conditions. In particular, it can be used to calculate $(h_x)_u$ and $(h_x)_d$ at the shock wave, as functions of time.

The initial condition $(h_x)_{t=0}$ is arbitrary and depends on the problem it is desired to study. Here, as an example, a channel flow which exhausts into a plenum with a given back pressure, such that a shock wave exists in the channel, is chosen. The flow is initially steady, but an unsteady disturbance in the back pressure has propagated to some position in the channel, say $x=\hat{x}=\hat{r}$ at $t=0$. The behavior of this "wave train" is then examined as it propagates upstream to the shock wave and induces oscillations in shock-wave position. The disturbance is written in terms of a variation in h_x from its steady-state solution $(h_x)_{ss}$. For example

$$h_r(r,0) = (h_x)_{ss} + \epsilon \sin a(r-\hat{r}) \;\text{ for } r>\hat{r} \tag{51a}$$

$$= (h_x)_{ss} \quad r \le \hat{r} \tag{51b}$$

Such variations are equivalent to an unsteady perturbation in the channel back pressure to order E^2, since the static pressure in the channel can be written as

$$P = 1 - E\gamma\left(\frac{2}{(\gamma+1)}f(x) + c_w\right)^{1/2} - E^2[\gamma f''(x)\frac{y^2}{2} + \gamma h_x] + \ldots \tag{52}$$

for this flow.

The solution for $(h_x)_{ss}$ is obtained from Eq. (39a) with $h_t=0$ and $A(t)$=constant, this constant having different values upstream and downstream of the shock wave. For the problem under consideration, the flow upstream of the shock is steady and $c_w=0$; hence it can be shown[6] that $A_u=0$. Downstream of the shock, A is determined from the equivalent steady-state shock jump condition [Eq. (36) with $u_{s2}=0$] to be

$$A_d = -\frac{2}{3}\gamma u_{lu}^3 \tag{53}$$

For the initial condition, x is replaced by r in $(h_x)_{ss}$. Now the equation for $(h_x)_{ss}$ can be applied anywhere in the channel

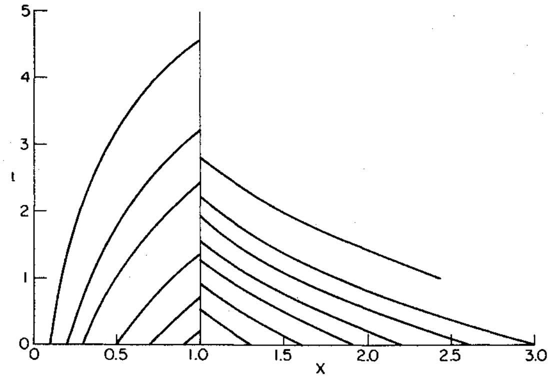

Fig. 2 x-t diagram showing characteristics for a parabolic wall; shock wave at $x=1$.

for $x > X_s$ when Eq. (53) is used for A_d. If it is applied at some x_b where the back pressure P_b is specified, so that $[h_x(x_b)]_{ss}$ is given, then this equation can be solved for $u_{1u} = u_{1u}(x_0)$ and thus for x_0, the steady-state shock location. For a given wall contour, then, one can find a curve of x_0 vs P_b. In a similar manner, the initial value of x_2 is set by the term of order E^3 in P_b. This term is assumed to be chosen such that $x_2(0) = 0$.

The fact that the shock-wave location is found by prescribing terms of order E^2 might at first appear questionable since for weak shocks entropy changes are of order E^3. This question has been addressed for steady flows in Ref. 6, where the more familiar condition of conservation of mass across the shock wave, for given downstream conditions, was used to position the shock. It was found that because, in transonic flow, each term of the expansion for ρU involves only lower-order terms of ρ and U, the shock could be located using only second-order terms. The same result can be shown to hold for the unsteady-flow problem considered here.[9]

V. Numerical Results

For a given wall shape, h_x is found by numerical integration of Eq. (47) along characteristics as follows. For a given characteristic, r = constant, s is found for each x from Eq. (48) using $u_1(x)$ as found from Eq. (29). Also, s is given for any t by Eq. (42). Therefore, one can find the s and r corresponding to each x and t, and thus any function of x and t can be evaluated for chosen values of r and s and integrated along the characteristics; in particular, $h_x(x,t)$ can be found using Eq. (47). Having found h_x, one can find u_2 and v_2 from Eqs. (16). Finally, U and V are given by Eqs. (38) and the shock velocity is found from Eq. (36), after obtaining h_x at x_0 as a function of time (i.e., h_{xd} and h_{xu}). In general, one would find the time at which a given characteristic downstream of the shock intercepts the shock wave ($x = x_0$) and then find the characteristic from upstream which reaches $x = x_0$ at the same time. Then, computing h_x along each of those characteristics would result in obtaining h_{xd} and h_{xu} at the same time, as required. For the present case where the flow upstream of the wave is steady, and h_{xu} is constant, such calculations are unnecessary. Following the calculation of u_{s2} from Eq. (36), x_2 is found from

$$x_2(t) = \int_0^t u_{s2} \mathrm{d}t \tag{54}$$

Figures 3 and 4 show numerical results obtained for a sinusoidal wave train imposed downstream of a shock wave in a symmetric nozzle-like channel which terminates in a constant-area section. Thus, the initial conditions are as given in Eqs. (51). The case where the velocity is sonic ($c_w = 0$) at the nozzle throat ($x = 0$) is considered. The geometry of the channel is shown in Fig. 3a. The channel is parabolic up to $x = 1$, with constant area for $x > 2$, and with a transition section for $1 < x < 2$. With an initial steady-state shock location at $x_0 = 1.5$, a wave train beginning at $\hat{x} = \hat{r} = 2.5$ is imposed in the flow at $t = 0$.

Figure 3a shows lines of constant velocity at $t = 0$. The two-dimensional nature of the flow is clearly illustrated. Underneath the half channel, in Fig. 3a, is shown the variation of the pressure from its sonic value, along the centerline of the channel. The position of the advancing wave front ($x = 2.5$) and the form of the pressure disturbance are also shown. In Fig. 3b, $t = 0.4$ and the wave train has moved upstream to $x = 1.85$, resulting in a modification of the lines of constant velocity. In figure 3c, $t = 0.6$ and the wave train has reached the shock wave, so that the entire flow field downstream of the shock is influenced by the wave train and is time dependent. At this time, the shock begins to move. The wave train imposed initially gives a pressure downstream of the shock wave lower than the steady-state value, so the shock begins to

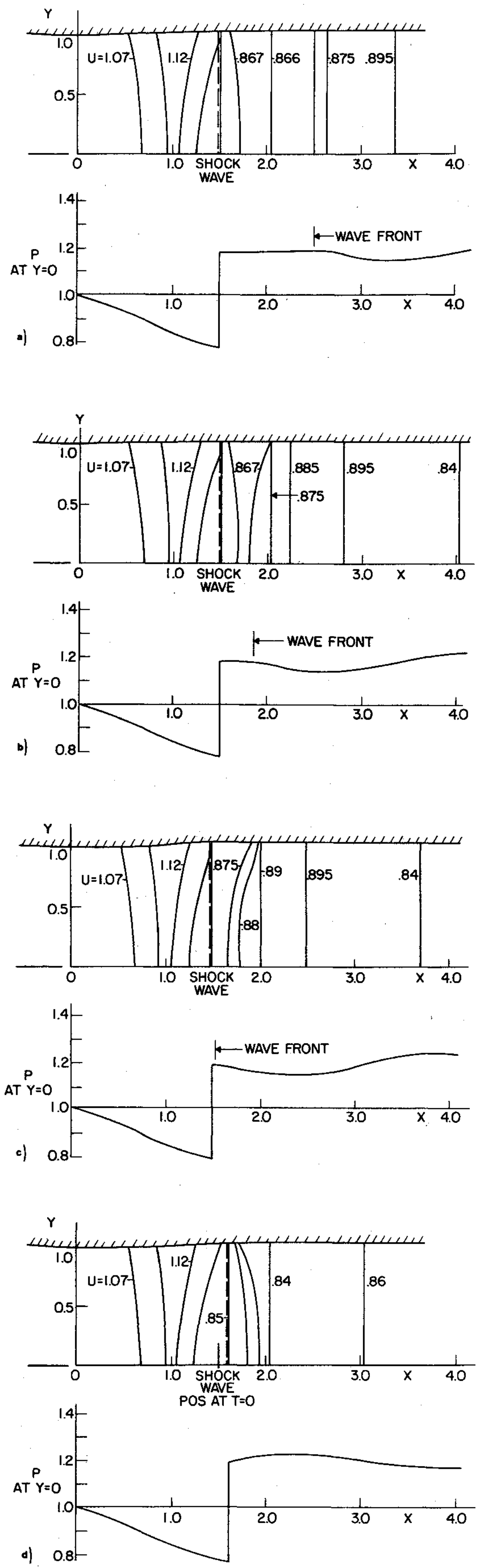

Fig. 3 Symmetric channel flow, showing lines of constant velocity, shock location, position of flow disturbance, and pressure distribution. $f(x) = 1.5x^2$ for $x \leq 1$; $f(x) = -1.5(x-2)^2 + 3$ for $1 < x < 2$; and $f(x) = 3$ for $x \geq 2$. Initial conditions as in Eq. (51a) with $\epsilon = 3$, $a = 2$, $\hat{r} = 2.5$. Steady-state shock position, $x_0 = 1.5$, $k = (\gamma + 1)/2, \gamma = 1.4, E = 0.1$. a) $t = 0$; b) $t = 0.4$; c) $t = 0.6$; d) $t = 1.6$.

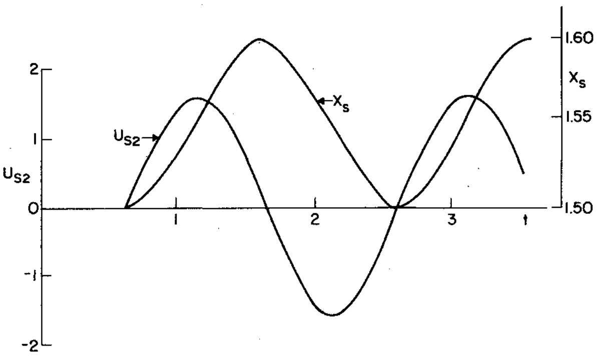

Fig. 4 Shock wave velocity and position as functions of time for conditions as given in Fig. 3.

move downstream, then moves upstream as the wave front brings higher pressures. It should be noted that the pressure distribution downstream of the shock, although modified by the channel shape, is essentially that associated with the pressure disturbance imposed on the flow and bears little resemblance to the steady-state distribution.

The shock wave continues to move downstream until $t=1.6$, when, for the parameter chosen, the minimum-pressure point reaches the shock wave and the increasing pressure causes the shock to move back upstream. Conditions at this time, when the shock reaches its maximum displacement ($x_h = 1.595$), are shown in Fig. 3d.

The variation in shock wave velocity, u_{s2}, in response to the imposed unsteady disturbance, and the corresponding variation in shock wave position are shown in Fig. 4. As noted previously, no shock motion starts until the wave train reaches the shock wave, at $t=0.6$. Additional examples may be found in Ref. 9.

VI. Conclusions

The analytical procedures described in this paper give insight into the nature of unsteady transonic flow in two-dimensional channels without recourse to more complicated numerical solutions. It is shown that the Szaniowski type of solution can be extended to unsteady flows and therefore has general utility in the analysis of transonic flow. Thus, the methods described in this paper permit consideration of the direct problem with arbitrary initial conditions, a result impossible to achieve with the similarity solutions used heretofore.

It is interesting to note that although variations in u_{s2}, and thus h_{xd}, in Eq. (36) are of order one, the resulting variation in shock-wave location is of order E. That is, since the steady-state shock location is found by setting $(h_x)_{ss}$ at some downstream location (i.e., by setting the back pressure) it would follow that a change in h_x of order one would result in a change in x_0 of order one. In the present case the variation in shock location is of order E because the variations in h_{xd} in Eq. (36) are oscillatory and balanced by a shock velocity, u_{s2} which is oscillatory so that the magnitude of x_2 never becomes large compared to one. Thus, there is a limit on the analysis; the magnitude and period of the impressed oscillations must be such that $x_2 = 0(1)$.

The method of solution described here for symmetric channels has also been extended to flow in asymmetric channels,[9] again with radius of curvature of order E^{-2}. The solutions yield basically the same results as those for symmetric channels, except that there is a difference between the pressure distributions along the upper and lower walls, and the lines of constant velocity are altered by the asymmetry.

Finally, it should be noted that the so-called slowly varying time regime, considered in this paper, covers a range of characteristic times, $\bar{T}_{ch}$, which is of considerable technical importance. Thus, if $k=1$, $E=0.1$, $\bar{a}^* \approx 1000$ ft/sec, and $\bar{L}$ is in the range of 0.1-1 ft, then $\bar{T}_{ch}$ is in the range of 10^{-3}-10^{-2} sec; in terms of a frequency, f (cycles/sec), then $100 < f < 1000$.

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